Algebraically expandable classes of implication algebras

Miguel Campercholi

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Abstract

An algebraically expandable class is a class of algebras axiomatizable by a set of sentences of the form $\forall\exists! \wedge p = q$. We find all algebraically expandable classes in the variety of implication algebras. A representation result for finite implication algebras is proved, and we also give a characterization of the class of congruence permutable implication algebras.

Keywords: Implication algebra; Global subdirect products; Algebraically expandable classes; Congruence permutability.

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1 Introduction and preliminaries

In this work we solve the following problem:

Characterize the subclasses of implication algebras that can be axiomatized by sentences of the form $\forall\exists! \wedge p = q$.

In the process we obtain a representation result for finite implication algebras, and as a by-product of our solution a number of interesting classes of implication algebras arise. We also obtain a characterization of the congruence permutable implication algebras.

Implication algebras, also known as Tarski algebras, have been introduced and studied by J. C. Abbott in [**?**], [**?**]. They are the $\{\rightarrow\}$-subreducts of Boolean algebras. It is also known that implication algebras are the algebraic counterpart of the implicational fragment of classical propositional logic [**?**].

For basic notions and notation in universal algebra and model theory we refer the reader to [**?**] and [**?**], respectively. Throughout this work we apply the notational convention that an algebra $\mathbf{A}$ has universe A, and we write $Con(\mathbf{A})$ to denote the set of all congruence relations of $\mathbf{A}$.

An *implication algebra* is an algebra $(L, \rightarrow, 1)$ satisfying:

(I_1) $x \rightarrow x \approx 1$,

(I_2) $(x \rightarrow y) \rightarrow x \approx x$,

(I_3) $x \to (y \to z) \approx y \to (x \to z)$,

(I_4) $(x \to y) \to y \approx (y \to x) \to x$.

We write $\mathcal{I}$ to denote the variety of implication algebras. The algebra $\mathbf{2} = (\{0,1\}, \to, 1)$, where $x \to y = 0$ iff $x = 1$ and $y = 0$, is the only (up to isomorphism) subdirectly irreducible in $\mathcal{I}$. Every implication algebra $\mathbf{L}$ is a join semi-lattice with respect to the ordering

$$x \leq y \Leftrightarrow x \to y = 1.$$

This ordering has 1 as its top element, and for $a, b \in L$ the join of a and b in $(L, \leq)$ is given by $a \vee b = (a \to b) \to b$. Also, for every $a \in L$, the interval $[a, 1]$ is boolean with respect to the $\leq$ ordering. The meet of $b, c \in [a, 1]$ is $(b \to (c \to a)) \to a$, and the complement of b relative to a is $b \to a$.

An *equational function definition sentence* (EFD-sentence) is a sentence of the form

$$\forall x_1 ... x_n \exists! z_1 ... z_m \; \varepsilon(\overline{x}, \overline{z}),$$

where $n \geq 0$, $m \geq 1$, and $\varepsilon(\overline{x}, \overline{z})$ is a conjunction of equations. It is convenient to introduce two first order sentences associated to an EFD-sentence. Let φ denote the sentence in the display above, then its uniqueness part is

$$U(\varphi) = \forall x_1 ... x_n \forall y_1 ... y_m \forall z_1 ... z_m \; (\varepsilon(\overline{x}, \overline{y}) \;\&\; \varepsilon(\overline{x}, \overline{z})) \Rightarrow \overline{y} = \overline{z},$$

and its existential part is

$$E(\varphi) = \forall x_1 ... x_n \exists z_1 ... z_m \; \varepsilon(\overline{x}, \overline{z}).$$

We say that an algebra $\mathbf{A} \in \mathcal{I}$ satisfies φ (in symbols $\mathbf{A} \vDash \varphi$) if and only if $\mathbf{A}$ satisfies $E(\varphi) \& U(\varphi)$. Observe that for every algebra $\mathbf{A}$ that satisfies φ we can define a function $[\varphi]^{\mathbf{A}} : A^n \to A^m$ by

$$[\varphi]^{\mathbf{A}}(\vec{a}) = \text{the only } \vec{b} \in A^m \text{ such that } \mathbf{A} \vDash \varepsilon(\vec{a}, \vec{b}).$$

If $\pi_j : A^m \to A$ is the jth canonical projection we write $[\varphi]^{\mathbf{A}}_j$ to denote $\pi_j \circ [\varphi]^{\mathbf{A}}$, for $j = 1, ..., m$. We refer to the functions $[\varphi]^{\mathbf{A}}$ and $[\varphi]^{\mathbf{A}}_j$ as the *functions defined by φ in* $\mathbf{A}$.

We call the structure $(\mathbf{A}, [\varphi]^{\mathbf{A}}_1, ..., [\varphi]^{\mathbf{A}}_n)$ an *algebraic expansion* of $\mathbf{A}$, since the new operations are defined as unique solutions to systems of equations. If a class $\mathcal{C}$ of algebras of the same language satisfies an EFD-sentence, then every algebra in $\mathcal{C}$ can be expanded in this way, thus, we call a class axiomatizable by EFD-sentences an *algebraically expandable* class. A study of this kind of classes for several other varieties can be found in [**?**].

2 Global representation for finite implication algebras

In this section we prove that every finite implication algebra is a global subdirect product where every factor is either the reduct of a finite boolean algebra without its bottom element, or the two-element implication algebra. It is worth mentioning that the technique developed by Vaggione and Gramaglia in [**?**] to establish a class of factors does not work for implication algebras. In this case we employed the aid of a computer to provide a candidate for a suitable class of factors. The result obtained only applies to finite implication algebras, and though it suffices for us to solve the problem stated at the beginning of this paper, we think it would be interesting if the representation result could be generalized to the whole variety $\mathcal{I}$.

A global subdirect product is a subdirect product with some additional requirements (see [**?**] for a definition). The proof of our representation result will follow from a property about congruence systems in finite implication algebras (see Lemma 2).

In the proof of Lemma 2 we make use of several well known facts about finite implication algebras. Let us start by summarizing these.

It is convenient to represent elements and congruences of a finite implication algebra as sets of co-atoms. (A similar approach is employed in [**?**], although to study a different problem.)

Let $\mathbf{L}$ be a finite implication algebra. A *co-atom* of $\mathbf{L}$ is an element in L covered by 1. Let

$$C(\mathbf{L}) = \{c \in L : c \text{ is a co-atom of } \mathbf{L}\}.$$

The map $C : L \to \mathcal{P}(C(\mathbf{L}))$, given by $C(x) = \{c \in C(\mathbf{L}) : x \leq c\}$, is an embedding from $\mathbf{L}$ into $(\mathcal{P}(C(\mathbf{L})), \to^d, \emptyset)$, where $A \to^d B = A^c \cap B$. The inverse of this embedding is given by $X \mapsto \wedge X$. For $\theta \in Con(\mathbf{L})$ let

$$C_\theta = \{c \in C(\mathbf{L}) : (c, 1) \in \theta\}.$$

The map $\theta \mapsto C_\theta$ is an isomorphism between $(Con(\mathbf{L}), \cap, \vee, \Delta, \nabla)$ and $(\mathcal{P}(C(\mathbf{L})), \cap, \cup, \emptyset, C(\mathbf{L}))$. The inverse of this map is given by $X \mapsto \bigvee_{c \in X} \theta(c, 1)$. Furthermore, for $a, b \in L$ we have

$$(a, b) \in \theta \Leftrightarrow C(a) - C_\theta = C(b) - C_\theta.$$

Next we define a class of finite implication algebras that play a key rôle in our work. For $n \geq 2$ let

$$F_n = \{0, 1\}^n - \{(0, ..., 0)\},$$

and let $\mathbf{F}_n$ be the subalgebra of $\mathbf{2}^n$ whose universe is F_n. For notational purposes it will be convenient to define $\mathbf{F}_1 = \mathbf{2}$. Let

$$\mathcal{F} = \{\mathbf{F}_n : n \geq 1\}.$$

In other words, $\mathcal{F}$ is the class containing the implicational reducts of one of each finite Boolean algebra without its bottom element, plus the implication algebra $\mathbf{2}$.

The lemma below gives sufficient conditions for a $\theta \in Con(\mathbf{L})$ to have the property $\mathbf{L}/\theta \cong \mathbf{F}_n$, for some $n \geq 2$.

Lemma 1 *Let* $\mathbf{L}$ *be a finite implication algebra. Suppose there is* $M \subseteq C(\mathbf{L})$ *such that:*

(a) $|M| = n \geq 2$,
(b) the meet of every proper subset of M *exists in* $\mathbf{L}$*, and*
(c) the meet of M *does not exist in* $\mathbf{L}$.

Then the congruence

$$\theta = \bigvee_{c \in C(\mathbf{L})-M} \theta(c,1),$$

is such that $\mathbf{L}/\theta \cong \mathbf{F}_n$.

Proof. Let $M = \{c_1, ..., c_n\} \subseteq C(\mathbf{L})$, and suppose M satisfies *(a)*, *(b)* and *(c)*. Let θ be as in the statement of this lemma. For $j = 1, ..., n$, let $\gamma_j : L \to \{0,1\}$ defined by

$$\gamma_j(x) = \begin{cases} 1 & \text{if } x \not\leq c_j \\ 0 & \text{if } x \leq c_j. \end{cases}$$

Define $F : L \to \{0,1\}^n$ by $F(x) = (\gamma_1(x), ..., \gamma_n(x))$. Note that

$$F(a) = F(b) \Leftrightarrow C(a) \cap M = C(b) \cap M,$$

and hence $\ker(F) = \theta$. Also note that, as M satisfies *(b)* and *(c)*, we have

$$F(L) = \{0,1\}^n - \{(0, ..., 0)\}.$$

Thus, $\mathbf{L}/\theta \cong \mathbf{F}_n$. ■

Given an implication algebra $\mathbf{L}$ define

$$\Sigma_{\mathbf{L}} = \{\theta \in Con(\mathbf{L}) : \text{there is } \mathbf{F} \in \mathcal{F} \text{ such that } \mathbf{L}/\theta \cong \mathbf{F}\}.$$

A system $(\theta_1, ..., \theta_n; a_1, ..., a_n)$ where $\theta_1, ..., \theta_n \in Con(\mathbf{L})$, $a_1, ..., a_n \in L$ is a *system of congruences in* $\mathbf{L}$ if $(a_i, a_j) \in \theta_i \vee \theta_j$, for $i, j = 1, ..., n$. An element $b \in L$ is a *solution* to the system $(\theta_1, ..., \theta_n; a_1, ..., a_n)$ when $(b, a_i) \in \theta_i$, for $i = 1, ..., n$.

Lemma 2 *Let* $\mathbf{L}$ *be a finite implication algebra. Let* $(\theta_1, ..., \theta_n; a_1, ..., a_n)$ *be a system of congruences in* $\mathbf{L}$ *such that for each* $\theta \in \Sigma_{\mathbf{L}}$ *there is some* $\theta_i \subseteq \theta$. *Then the system has a solution in* L.

Proof. Let us start by translating the condition of $(\theta_1, ..., \theta_n; a_1, ..., a_n)$ being a system into terms of co-atoms. We know that

$$(a_i, a_j) \in \theta_i \vee \theta_j, \text{ for } i, j = 1, .., n.$$

Equivalently

$$C(a_i) - (C_{\theta_i} \cup C_{\theta_j}) = C(a_j) - (C_{\theta_i} \cup C_{\theta_j}), \text{ for } i, j = 1, .., n. \quad (*)$$

Let

$$B = \bigcup_{i=1}^{n} C(a_i) - C_{\theta_i}.$$

We prove that $b = \wedge B$ exists in $\mathbf{L}$, and that b is a solution for the system $(\theta_1, ..., \theta_n; a_1, ..., a_n)$. For the sake of contradiction suppose $\wedge B$ does not exist. Then, there is a minimal subset $M \subseteq B$ that has no meet in $\mathbf{L}$. Note that every proper subset of M has a meet in $\mathbf{L}$. So, if we define

$$\theta = \bigvee_{c \in C(\mathbf{L}) - M} \theta(c, 1),$$

by Lemma 1, we have $\theta \in \Sigma_{\mathbf{L}}$. Thus, there is $k \in \{1, ..., n\}$ such that $\theta_k \subseteq \theta$. This is the same as to say that $C_{\theta_k} \subseteq C_\theta = C(\mathbf{L}) - M$. Now, $M \cap C_{\theta_k} = \emptyset$ and it follows that $M \subseteq B - C_{\theta_k}$. Observe that

$$B - C_{\theta_k} = \bigcup_{i=1}^{n} C(a_i) - (C_{\theta_i} \cup C_{\theta_k}),$$

and due to (*),

$$\begin{aligned} B - C_{\theta_k} &= \bigcup_{i=1}^{n} C(a_k) - (C_{\theta_i} \cup C_{\theta_k}) \\ &= C(a_k) - C_{\theta_k} \\ &\subseteq C(a_k). \end{aligned}$$

So, we have that $M \subseteq C(a_k)$. In other words, M is included in the interval $[a_k, 1]$, which we know is boolean. Thus M has a meet in $\mathbf{L}$, arriving at a contradiction. It follows that B must have a meet b. To conclude we show that b is a solution to the system $(\theta_1, ..., \theta_n; a_1, ..., a_n)$. This is equivalent to

$$B - C_{\theta_i} = C(a_i) - C_{\theta_i}, \text{ for } i = 1, ..., n.$$

Let $j \in \{1, ..., n\}$. Using elementary set algebra and (*) we have:

$$\begin{aligned} B - C_{\theta_i} &= \bigcup_{i=1}^{n} C(a_i) - (C_{\theta_i} \cup C_{\theta_j}) \\ &= \bigcup_{i=1}^{n} C(a_j) - (C_{\theta_i} \cup C_{\theta_j}) \\ &= C(a_j) - C_{\theta_j}. \end{aligned}$$

■

We are now ready to prove our representation result.

Proposition 3 *Every finite member of $\mathcal{I}$ is isomorphic to a global subdirect product with factors in $\mathcal{F} = \{\mathbf{F}_n : n \geq 1\}$.*

Proof. This follows from Lemma 2 and [?]. ■

The general definition of when an algebra is globally indecomposable [?] is somewhat involved, but it turns out that for the finite case it can be stated as follows. A finite algebra $\mathbf{A}$ is *globally indecomposable* if every decomposition of $\mathbf{A}$ as a global subdirect product has a factor isomorphic to $\mathbf{A}$.

Corollary 4 *Every algebra in $\mathcal{F}$ is globally indecomposable.*

Proof. As $\mathbf{2}$ is subdirectly irreducible it must be globally indecomposable. Observe that $\mathbf{F}_n$ is not a homomorphic image of $\mathbf{F}_{n+1}$, for $n \geq 2$. So, Proposition 3 says that if $\mathbf{F}_n$ $(n \geq 2)$ has a global representation, then every one of its factors must be isomorphic to $\mathbf{2}$. It is not difficult to see that this is not possible. ■

The above corollary says that our class $\mathcal{F}$ of factors is as small as possible, and the representation theorem is *a la* Birkhoff. As explained in the beginning of this section, a suitable class of factors that globally represents the whole variety remains unknown to us.

3 Algebraically expandable classes in $\mathcal{I}$

In this section we characterize all algebraically expandable classes in $\mathcal{I}$. Our first step will be to prove that the algebraically expandable classes of $\mathcal{I}$ are in correspondence with the algebraically expandable classes of $\mathcal{F}$ (Lemma 8). The rôle of the members of $\mathcal{F}$ in the study of axiomatizability by EFD-sentences in $\mathcal{I}$ closely resembles the rôle subdirectly irreducibles would play in studying axiomatizability by identities in a given variety. One of the reasons to this is the following:

Lemma 5 ([?]) *Suppose $\mathbf{A} \subseteq \Pi\{\mathbf{A}_i : i \in I\}$ is a global subdirect product, and let φ be an EFD-sentence. If $\mathbf{A}_i \vDash \varphi$, for all $i \in I$ then $\mathbf{A} \vDash \varphi$.*

Though in our case we do not have a representation result for all implication algebras, we can circumvent this difficulty thanks to the next lemma.

Lemma 6 ([?]) *Let $\mathcal{Q}$ be a quasi-variety, and let φ and ψ be EFD-sentences in the language of $\mathcal{Q}$. Assume that $Mod(U(\varphi)) \cap \mathcal{Q}$ and $Mod(U(\psi)) \cap \mathcal{Q}$ are finitely generated quasi-varieties. Suppose that for every finite $\mathbf{A} \in \mathcal{Q}$ we have that $\mathbf{A} \vDash \varphi$ implies $\mathbf{A} \vDash \psi$. Then $Mod(\varphi) \cap \mathcal{Q} \subseteq Mod(\psi) \cap \mathcal{Q}$.*

The following easy observation is part of several proofs in the sequel, so it is convenient to state it as a separate lemma.

Lemma 7 *Let φ be an EFD-sentence with a non-trivial model in $\mathcal{I}$. Then $\mathbf{2} \vDash \varphi$ and $\mathcal{I} \vDash U(\varphi)$.*

Proof. Let $\mathbf{L} \in Mod(\varphi) \cap \mathcal{I}$ be non-trivial. Note that $\mathbf{2}$ is a homomorphic image of $\mathbf{L}$ and as $E(\varphi)$ is a positive sentence it follows that $\mathbf{2} \vDash E(\varphi)$. Since $\mathbf{L}$ has a subalgebra isomorphic to $\mathbf{2}$ and $U(\varphi)$ is a quasi-identity we have $\mathbf{2} \vDash U(\varphi)$. Now, every member of $\mathcal{I}$ is isomorphic to a subdirect power of $\mathbf{2}$, so $\mathcal{I} \vDash U(\varphi)$, as subdirect products preserve quasi-identities. ∎

Now we can prove that the class $\mathcal{F}$ has the desired property.

Lemma 8 *Let φ, ψ be EFD-sentences. Then,*

$$Mod(\varphi) \cap \mathcal{F} \subseteq Mod(\psi) \cap \mathcal{F} \Rightarrow Mod(\varphi) \cap \mathcal{I} \subseteq Mod(\psi) \cap \mathcal{I}.$$

Proof. Let φ, ψ be EFD-sentences such that $Mod(\varphi) \cap \mathcal{F} \subseteq Mod(\psi) \cap \mathcal{F}$. Due to Lemma 6 it suffices to show that every finite model of φ in $\mathcal{I}$ is a model of ψ. Let $\mathbf{L} \in Mod(\varphi) \cap \mathcal{I}$ be finite. If $\mathbf{L}$ is trivial it is obvious that $\mathbf{L} \vDash \psi$, so assume $\mathbf{L}$ is non-trivial. From Lemma 7 we obtain $\mathbf{2} \in Mod(\varphi) \cap \mathcal{F} \subseteq Mod(\psi) \cap \mathcal{F}$ and $\mathcal{I} \vDash U(\psi)$. By Proposition 3 we know that $\mathbf{L}$ is isomorphic to a global subdirect product $\mathbf{G}$ with factors in $\mathcal{F}$. Let $S \subseteq \mathcal{F}$ be the set factors of $\mathbf{G}$; note that each member of S is a homomorphic image of $\mathbf{G}$. Thus, $S \vDash E(\varphi)$ and we already proved that $\mathcal{I} \vDash U(\varphi)$, so we have $S \subseteq Mod(\varphi) \cap \mathcal{F} \subseteq Mod(\psi) \cap \mathcal{F}$. Finally, apply Lemma 5 to obtain that $\mathbf{L} \vDash \psi$. ∎

Our next step is to understand axiomatizability by EFD-sentences within the class $\mathcal{F}$. The main tools in this process are the functions associated to each EFD-sentence, and their properties. (The definition of the functions defined by an EFD-sentence can be found in the first section.)

Lemma 9 ([?]) *Let φ be an EFD-sentence.*

1. *Let $\{\mathbf{A}_i : i \in I\}$ be a family of algebras such that $\mathbf{A}_i \vDash \varphi$, for all $i \in I$, and let $\mathbf{P} = \prod_{i \in I} \mathbf{A}_i$. Then $\mathbf{P} \vDash \varphi$ and*

$$[\varphi]_j^{\mathbf{P}}(p_1, ..., p_n)(i) = [\varphi]_j^{\mathbf{A}_i}(p_1(i), ..., p_n(i)),$$

   for every $i \in I$, $j = 1, ..., m$, and any $p_1, ..., p_n \in \mathbf{P}$.

2. *Suppose $\mathbf{A} \vDash \varphi$, and let $\mathbf{B}$ be a subalgebra of $\mathbf{A}$. Then $\mathbf{B} \vDash \varphi$ iff $[\varphi]^{\mathbf{A}}(B^n) \subseteq B^m$.*

Lemma 10 *Let φ be an EFD-sentence. Then for every $n \geq 1$*

$$\mathbf{F}_{n+1} \vDash \varphi \Rightarrow \mathbf{F}_n \vDash \varphi.$$

Proof. The case $n = 1$ is already taken care of by Lemma 7, so assume $n \geq 2$. For the sake of contradiction suppose there is an EFD-sentence φ such that $\mathbf{F}_{n+1} \vDash \varphi$ and $\mathbf{F}_n \nvDash \varphi$. As, by Lemma 7, we have $\mathcal{I} \vDash U(\varphi)$, it follows that $\mathbf{F}_n \nvDash E(\varphi)$. Moreover, note that since $\mathbf{F}_n$ is generated by $\{(1, 0, ..., 0), ..., (0, ..., 0, 1)\}$, we can assume w.l.o.g. that

$$\varphi = \forall x_1 ... x_n \exists! z_1 ... z_m \; \varepsilon(\overline{x}, \overline{z}),$$

and that a solution $\vec{b} \in (F_n)^m$ fails to exist for $x_1 = (1, 0, ..., 0), ..., x_n = (0, ..., 0, 1)$. Since $\mathbf{F}_{n+1}$ is non-trivial Lemma 7 implies $\mathbf{2} \models \varphi$, and (2) from Lemma 9 says that $\mathbf{2}^{n+1} \models \varphi$. Next, note that the map $\gamma : F_n \to F_{n+1}$, defined by $\gamma(a_1, ..., a_n) = (a_1, ..., a_n, 1)$ is an embedding. Let $\bar{F}_n = \gamma(F_n)$. By our assumptions about φ, we know that

$$[\varphi]^{\mathbf{2}^{n+1}}((1, 0, ..., 0, 1), (0, 1, 0, ..., 0, 1), ..., (0, ..., 0, 1, 1)) \notin (\bar{F}_n)^m.$$

Let $\bar{\mathbf{2}}^n$ be the subalgebra of $\mathbf{2}^{n+1}$ whose universe is $\{(a_1, ..., a_n, 1) : a_i \in \{0, 1\}\}$. Clearly $\bar{\mathbf{2}}^n$ and $\mathbf{2}^n$ are isomorphic, and hence $\bar{\mathbf{2}}^n \models \varphi$. Since $\bar{F}_n \subseteq \{(a_1, ..., a_n, 1) : a_i \in \{0, 1\}\}$, applying (2) of Lemma 9 yields

$$[\varphi]^{\mathbf{2}^{n+1}}((\bar{F}_n)^n) \subseteq \{(a_1, ..., a_n, 1) : a_i \in \{0, 1\}\}^m.$$

Combining the facts in the two displays above we obtain

$$[\varphi]_j^{\mathbf{2}^{n+1}}((1, 0, ..., 0, 1), (0, 1, 0, ..., 0, 1), ..., (0, ..., 0, 1, 1)) = (0, ..., 0, 1),$$

for some $j \in \{1, ..., m\}$. Now, (1) of Lemma 9 says that the coordinate functions of $[\varphi]_j^{\mathbf{2}^{n+1}}$ are all equal to $[\varphi]_j^{\mathbf{2}}$, and thus

$$\begin{aligned}
[\varphi]_j^{\mathbf{2}}(1, 0, ..., 0) &= 0 \\
[\varphi]_j^{\mathbf{2}}(0, 1, 0, ..., 0) &= 0 \\
&\vdots \\
[\varphi]_j^{\mathbf{2}}(0, ..., 0, 1) &= 0 \\
[\varphi]_j^{\mathbf{2}}(1, 1, ..., 1) &= 1.
\end{aligned}$$

But this produces

$$\begin{aligned}
&[\varphi]_j^{\mathbf{2}^{n+1}}((1, 0, ..., 0, 0), (0, 1, 0, ..., 0), ..., (0, ..., 0, 1, 1)) = \\
&= ([\varphi]_j^{\mathbf{2}}(1, 0, ..., 0), [\varphi]_j^{\mathbf{2}}(0, 1, 0, ..., 0), ..., [\varphi]_j^{\mathbf{2}}(0, ..., 0, 1), [\varphi]_j^{\mathbf{2}}(0, ..., 0, 1)) \\
&= (0, 0, ..., 0),
\end{aligned}$$

which is a contradiction, as Lemma 9 implies

$$[\varphi]^{\mathbf{2}^{n+1}}((F_{n+1})^n) \subseteq (F_{n+1})^m.$$

■

Next we proceed to show that for each $n \geq 2$ there is an EFD-sentence φ_n such that $F_{n-1} \models \varphi_n$ and $F_n \not\models \varphi_n$. Let $n \geq 2$, and let $x_1, ..., x_n$ be variables. For $i = 1, .., n$ define the terms

$$s_i^n(x_1, ..., x_n) = \bigvee_{j=1,\ j \neq i}^{n} x_j,$$

and let

$$\varphi_n = \forall x_1 ... x_n \exists ! z \left(z \leq s_1^n(\overline{x}) \ \& \ ... \ \& \ z \leq s_n^n(\overline{x}) \ \& \ (\bigvee_{i=1}^{n} (s_i^n(\overline{x}) \to z)) = 1) \right).$$

Though φ_n and s_i^n are defined using the symbols the $\leq$ and $\vee$, these symbols are just shorthand, and it should be clear that φ_n is a sentence in the language $\{\to, 1\}$.

Lemma 11 *For $n \geq 2$ the sentence φ_n holds in $\mathbf{2}$, and the function defined by φ_n in $\mathbf{2}$ is the function f_n, defined by*

$$f_n(a_1, ..., a_n) = \begin{cases} 1 & \text{if } |\{a_i : a_i = 1\}| \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $a_1, ..., a_n \in \{0, 1\}$. Observe that if $|\{a_i : a_i = 1\}| \geq 2$, then $s_i(a_1, ..., a_n) = 1$, for $i = 1, ..., n$. Hence, the only $b \in \{0, 1\}$ that makes φ_n true for $x_1 = a_1, ..., x_n = a_n$ is 1. On the other hand, if $|\{a_i : a_i = 1\}| \leq 1$, then $s_j(a_1, ..., a_n) = 0$, for some j. Thus, the only $b \in \{0, 1\}$ that makes φ_n true for $x_1 = a_1, ..., x_n = a_n$ is 0. ■

Lemma 12 *For $n \geq 2$ we have*

1. $\mathbf{F}_{n-1} \vDash \varphi_n$, *and*
2. $\mathbf{F}_n \nvDash \varphi_n$.

Proof. (1). The case $n = 2$ is already proved in Lemma 11, so assume $n \geq 3$. By Lemma 11 and (1) of Lemma 9 we know that $\mathbf{2}^k \vDash \varphi_n$, for $k \geq 1$. Hence, by (2) of Lemma 9, to prove $\mathbf{F}_{n-1} \vDash \varphi_n$ we only need to check that $[\varphi_n]^{\mathbf{2}^{n-1}}((F_{n-1})^n) \subseteq F_{n-1}$. Let $a_1, ..., a_n \in F_{n-1} = 2^{n-1} - \{(0, ..., 0)\}$. The pigeonhole principle yields that for some $i \in \{1, ..., n-1\}$ and $j, k \in \{1, ..., n\}$, we have $a_j(i) = 1 = a_k(i)$. Now, (1) of Lemma 9 says that

$$[\varphi_n]^{\mathbf{2}^{n-1}}(a_1, ..., a_n) = ([\varphi_n]^{\mathbf{2}}(a_1(1), ..., a_n(1)), ..., [\varphi_n]^{\mathbf{2}}(a_1(n-1), ..., a_n(n-1)),$$

and Lemma 11 produces

$$[\varphi_n]^{\mathbf{2}^{n-1}}(a_1, ..., a_n) = (f_n(a_1(1), ..., a_n(1)), ..., f_n(a_1(n-1), ..., a_n(n-1)).$$

So, by the definition of f_n, at least one of the coordinates of $[\varphi_n]^{\mathbf{2}^{n-1}}(a_1, ..., a_n)$ is 1, and thus $[\varphi_n]^{\mathbf{2}^{n-1}}(a_1, ..., a_n) \in F_{n-1}$.

(2). In view of item (2) of Lemma 9 it suffices to find $a_1, ..., a_n \in F_n$ such that $[\varphi_n]^{\mathbf{2}^n}(a_1, ..., a_n) \notin F_n$. A coordinatewise computation as in the first part of this proof shows that

$$[\varphi_n]^{\mathbf{2}^n}((1, 0, ..., 0), (0, 1, 0, ..., 0), ..., (0, ..., 0, 1)) = (0, ..., 0).$$

■

We have completed our study of the class $\mathcal{F}$, and are now ready to prove our main result.

Let $\mathcal{I}_n$ be the class of algebras in $\mathcal{I}$ that satisfy φ_n, i.e.,

$$\mathcal{I}_n = Mod(\varphi_n) \cap \mathcal{I}.$$

Theorem 13 *If $\mathcal{C} \subseteq \mathcal{I}$ is an algebraically expandable class then either $\mathcal{C} = \{$trivial algebras in $\mathcal{I}\}$, $\mathcal{C} = \mathcal{I}$ or $\mathcal{C} = \mathcal{I}_n$ for some $n \geq 2$. Furthermore we have*

$$\{\textit{trivial algebras in } \mathcal{I}\} \subsetneq \mathcal{I}_2 \subsetneq \mathcal{I}_3 \subsetneq \cdots \subsetneq \mathcal{I}.$$

Proof. We prove that for any EFD-sentence φ, either $Mod(\varphi) \cap \mathcal{I} = \{$trivial algebras in $\mathcal{I}\}$, $Mod(\varphi) \cap \mathcal{I} = \mathcal{I}$, or there there is $n \geq 2$ such that $Mod(\varphi) \cap \mathcal{I} = Mod(\varphi_n) \cap \mathcal{I}$.

Let φ be an EFD-sentence. If $\mathbf{F}_n \vDash \varphi$, for all $n \geq 1$, then Lemma 8 implies that φ holds in every implication algebra. Suppose next that there is $m \geq 1$ such that $\mathbf{F}_m \nvDash \varphi$. Then, by Lemma 10, there is a smallest $n_0 \in \omega$ such that $\mathbf{F}_{n_0} \nvDash \varphi$. If $n_0 = 1$ then, $Mod(\varphi) \cap \mathcal{F} = \emptyset$ and Lemma 8 implies that $Mod(\varphi) \cap \mathcal{I} = \{$trivial algebras in $\mathcal{I}\}$. On the other hand, if $n_0 \geq 2$, it is clear that $Mod(\varphi) \cap \mathcal{F} = Mod(\varphi_{n_0}) \cap \mathcal{F}$, and Lemma 8 produces $Mod(\varphi) \cap \mathcal{I} = Mod(\varphi_{n_0}) \cap \mathcal{I}$.

Next we prove that $\mathcal{I}_n \subsetneq \mathcal{I}_{n+1}$, for $n \geq 2$. From Lemmas 10 and 12 we know that

$$Mod(\varphi_n) \cap \mathcal{F} = \{\mathbf{F}_1, ..., \mathbf{F}_{n-1}\} \subsetneq \{\mathbf{F}_1, ..., \mathbf{F}_n\} = Mod(\varphi_{n+1}) \cap \mathcal{F},$$

and Lemma 8 yields

$$\mathcal{I}_n = Mod(\varphi_n) \cap \mathcal{I} \subsetneq Mod(\varphi_{n+1}) \cap \mathcal{I} = \mathcal{I}_{n+1}.$$

Now it is quite easy to see that if $\mathcal{C} \subseteq \mathcal{I}$ can be axiomatized by EFD-sentences relative to $\mathcal{I}$, it must be one of the classes in the statement of this theorem. Suppose $\mathcal{C} = Mod(\Gamma) \cap \mathcal{I}$, for some set Γ of EFD-sentences. By what have proven so far we know that there are three possible kinds of sentences in Γ: the ones that only have trivial models in $\mathcal{I}$, the ones that are valid in $\mathcal{I}$, and the ones equivalent to some φ_n. If Γ contains a sentence of the first kind, then $\mathcal{C}$ is obviously the class of trivial algebras in $\mathcal{I}$. As we can safely assume that there are no sentences of the second kind in Γ, the only case remaining is the one in which Γ only contains sentences of the third kind. But this case follows easily from the fact that the classes $\mathcal{I}_n$ form a chain. ■

In the sequel we conduct a brief study of the $\mathcal{I}_n$ classes.

Lemma 14 *For $n \geq 2$ and $\mathbf{L} \in \mathcal{I}$ t.f.a.e.:*

1. $\mathbf{L} \vDash \varphi_n$.

2. *Let $S \subseteq L$ such that $|S| \leq n$, and suppose that every subset S with up to $n-1$ elements has a meet in $\mathbf{L}$. Then S has a meet in $\mathbf{L}$.*

3. *The meet $\bigwedge_{j=1}^{n} s_j^n(a_1, ..., a_n)$ exists in $\mathbf{L}$, for all $a_1, ..., a_n \in L$.*

Proof. (1)⇒(2). Let $S = \{s_1, ..., s_n\}$, and let $a_i = \wedge(S - \{s_i\})$, $i = 1, ..., n$. Take $b = [\varphi_n]^{\mathbf{L}}(a_1, ..., a_n)$. Note that

$$s_j^n(a_1, ..., a_n) = \bigvee_{i=1,\ i\neq j}^{n} a_i \leq s_i,$$

for $j = 1, ..., n$. Thus, by the definition of φ_n, it follows that $b \leq s_1, ..., b \leq s_n$, and as S has a lower bound it has a meet.

(2)⇒(1). In view of Lemmas 11 and 7 we only need to prove that $\mathbf{L} \vDash E(\varphi)$. Let $a_1, ..., a_n \in L$, and observe that $S = \{s_j^n(a_1, ..., a_n) : j = 1, ..., n\}$ satisfies the requirements stated in (2), and thus it has a meet $b \in L$. It is easy to check that $z = b$ satisfies $E(\varphi)$ for $x_1 = a_1, ..., x_n = a_n$.

(3)⇔(2). This is left to the reader. ■

We show next that each class $\mathcal{I}_n$ is actually the reduct of a variety to the language of $\mathcal{I}$. From item (3) of Lemma 14 we know that for every $\mathbf{L} \in \mathcal{I}_n$ a new n-ary operation $\mu_n : L^n \to L$ can be defined by

$$\mu_n(a_1, ..., a_n) = \bigwedge_{j=1}^{n} s_j^n(a_1, ..., a_n).$$

Now, extend the language of $\mathcal{I}$ with the n-ary function symbol μ_n, and define the following class of algebras in this new language:

$$\mathcal{M}_n = \{(\mathbf{L}, \mu_n) : \mathbf{L} \in \mathcal{I}_n\}.$$

Proposition 15 *The class $\mathcal{M}_n$ is a variety axiomatizable by*

(I$_1$), (I$_2$), (I$_3$), (I$_4$),

$\forall \overline{x}\ \mu_n(\overline{x}) \leq s_j^n(\overline{x})$, *for $j = 1, ..., n$,*

$\forall \overline{x}\ \bigvee_{i=1}^{n}(s_i^n(\overline{x}) \to \mu_n(\overline{x})) = 1.$

Furthermore, $Con(\mathbf{L}, \mu_n) = Con(\mathbf{L})$, for $\mathbf{L} \in \mathcal{I}_n$.

Proof. Suppose $(L, \to, 1, \rho)$ satisfies the above axioms. Clearly, $(L, \to, 1) \vDash E(\varphi_n)$ and, by Lemma 7, $(L, \to, 1) \vDash U(\varphi_n)$. Hence, we have $(L, \to, 1) \in \mathcal{I}_n$, and necessarily $\rho = \mu_n$. So, $(L, \to, 1, \rho) \in \mathcal{M}_n$. It is straightforward to show that if $\mathbf{L} \in \mathcal{M}_n$, then it satisfies the above identities. The furthermore part follows from the fact that whenever a meet exists in an implication algebra it can be written as a polynomial in the language $\{\rightarrow\}$, and thus the congruences of an implication algebra $\mathbf{L}$ preserve whatever meets exist $\mathbf{L}$. In particular, they preserve μ_n. ■

An immediate consequence of the above theorem is that the only (up to isomorphisms) subdirectly irreducible member of $\mathcal{M}_n$ is $(\mathbf{2}, \mu_n)$. Another consequence is that every finite member of $\mathcal{M}_n$ is isomorphic to a global subdirect product with factors in $\{\mathbf{F}_1, ..., \mathbf{F}_{n-1}\}$.

The class $\mathcal{I}_2$ is, of course, the class formed by the reducts of generalized boolean algebras to the language of $\mathcal{I}$. It also is the class of congruence permutable implication algebras. We conclude this work with a characterization of congruence permutable implication algebras.

Theorem 16 *Let* $\mathbf{L} \in \mathcal{I}$. *T.f.a.e.:*

1. $\mathbf{L}$ *is congruence permutable.*
2. *The meet of any two elements of* L *exists.*
3. $\mathbf{L} \models \forall x_1 x_2 \exists! z \; z \leq x_1 \;\&\; z \leq x_2 \;\&\; ((x_1 \rightarrow z) \vee (x_2 \rightarrow z) = 1)$.
4. $\mathbf{L}$ *is isomorphic to a global subdirect product whose factors are all isomorphic to* $\mathbf{2}$.

    Furthermore, if $\mathbf{L}$ *is finite the above are equivalent to:*

5. $\mathbf{L}$ *has no homomorphic images in* $\mathcal{F} - \{\mathbf{2}\}$.

Proof. (1)⇒(4). This implication follows from [**?**].

(4)⇒(3). Since $\mathbf{2}$ satisfies the EFD-sentence in (3), Lemma 5 yields this implication at once.

(3)⇒(2). Direct from Lemma 14.

(2)⇒(1). This is proved in [**?**].

To prove the furthermore part note that, due to Proposition 3, we have that (5) is equivalent to (4). ■

In the paper [**?**] the authors characterize when two given congruences of an implicative algebra permute, and also derive the equivalence of (1) and (2) in the above theorem.

camper@mate.uncor.edu
Facultad de Matemática, Astronomía y Física (Fa.M.A.F.)
Universidad Nacional de Córdoba - Ciudad Universitaria
5000 - Córdoba, ARGENTINA