Compact factor congruences implies Boolean factor congruences

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Abstract. We prove that any variety $\mathcal{V}$ in which every factor congruence is compact has *Boolean factor congruences,* i.e. for all A in $\mathcal{V}$ the set of factor congruences of A is a distributive sublattice of the congruence lattice of A.

A variety $\mathcal{V}$ has *compact factor congruences (CFC)* if every factor congruence of any algebra in $\mathcal{V}$ is compact. Examples of varieties with CFC are bounded semilattices and every variety with the Fraser-Horn property [3] in which no non trivial algebra has a trivial subalgebra. A variety $\mathcal{V}$ has *Boolean factor congruences (BFC)* if the set of factor congruences of any algebra in $\mathcal{V}$ is a distributive sublattice of its congruence lattice. In this paper we prove the following:

Theorem 1. *CFC implies BFC.*

Our proof is based on the fact that in varieties with CFC we have *central elements, i.e.* elements which define a direct decomposition of the algebra just like central idempotents in rings or complemented elements in bounded distributive lattices. Central elements were introduced in [6] and the reader can see [7], [8], [9] and [10] for other applications and basic results. For basic facts and notation on universal algebra and model theory the reader can consult [5] and [2].

For the remainder of the paper let $\mathcal{V}$ be a variety with CFC. For $A \in \mathcal{V}$ and $\vec{a}, \vec{b} \in A^n$, we will use $\theta^A(\vec{a}, \vec{b})$ to denote the congruence generated by the set $\{(a_k, b_k) : 1 \leq k \leq n\}$. We will use the following (Grätzer) version of the Maltsev key observation on principal congruences.

Lemma 2. *Let A be any algebra and let $a, b \in A$, $\vec{a}, \vec{b} \in A^n$. Then $(a, b) \in \theta^A(\vec{a}, \vec{b})$ if and only if there exist $n+m$-ary terms $P_1(\vec{x}, \vec{u}), ..., P_k(\vec{x}, \vec{u})$, with k odd and, $\vec{\lambda} \in A^m$ such that:*

$$\begin{aligned} a &= P_1(\vec{a}, \vec{\lambda}) \\ P_i(\vec{b}, \vec{\lambda}) &= P_{i+1}(\vec{b}, \vec{\lambda}),\ i \text{ odd} \\ P_i(\vec{a}, \vec{\lambda}) &= P_{i+1}(\vec{a}, \vec{\lambda}),\ i \text{ even} \\ P_k(\vec{b}, \vec{\lambda}) &= b \end{aligned}$$

Corollary 3. *For every homomorphism $F : A \to B$, if $(a, b) \in \theta^A(\vec{a}, \vec{b})$, then $(F(a), F(b)) \in \theta^B(F(\vec{a}), F(\vec{b}))$.*

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Given a set of variables X, we use $T(X)$ to denote the set of terms with variables in X. The $\mathcal{V}$-free algebra freely generated by X, will be denoted by $F(X)$. As usual, we will identify elements of $F(X)$ with terms where no possibility of confusion arises. The canonical projection on the i-th coordinate of a direct product will be denoted by π_i. If $\vec{a} \in A^n$ and $\vec{b} \in B^n$, then we will use $[\vec{a}, \vec{b}]$ to denote the n-tuple $((a_1, b_1), ..., (a_n, b_n)) \in (A \times B)^n$.

Our first step is to provide an adequate context to define the concept of central element.

Lemma 4. *There exist unary terms* $0_1(w), \ldots, 0_N(w)$, $1_1(w), \ldots, 1_N(w)$ *such that for every algebra* $A = A_1 \times A_2 \in \mathcal{V}$, $(\lambda_1, \lambda_2) \in A$,

$$\begin{aligned} \ker \pi_1 &= \theta^A\left([\vec{0}(\lambda_1), \vec{0}(\lambda_2)], [\vec{0}(\lambda_1), \vec{1}(\lambda_2)]\right) \\ \ker \pi_2 &= \theta^A\left([\vec{1}(\lambda_1), \vec{1}(\lambda_2)], [\vec{0}(\lambda_1), \vec{1}(\lambda_2)]\right) \end{aligned}$$

Proof: We note that by Corollary 3 we can suppose that $A = F(X) \times F(X)$, where X is a infinite set of variables. Since $\ker \pi_1$ is compact, there exist $\vec{B}, \vec{C}, \vec{D} \in T(X)^M$ such that

$\ker \pi_1 = \theta^A([\vec{B}, \vec{C}], [\vec{B}, \vec{D}])$

Note that, at expense of growing M, we can suppose $\vec{B} = \vec{C}$. Thus we have

$\ker \pi_1 = \theta^A([\vec{C}, \vec{C}], [\vec{C}, \vec{D}])$

Let $w \in X$ be a variable which does not occur in $\vec{C}, \vec{D}$. We will prove that

(*) $\ker \pi_1 = \theta^A\left([\vec{C}(w, ..., w), \vec{C}(w, ..., w)], [\vec{C}(w, ..., w), \vec{D}(w, ..., w)]\right)$

Take $p, q, r \in F(X)$. Let $x, y, z \in X - \{w\}$ be distinct variables which do not occur in $\vec{C}, \vec{D}$. Let $h : F(X) \to F(X)$, be the homomorphism given by the prescriptions

$h(x) = p$

$h(y) = q$

$h(z) = r$

$h(u) = w$, for every $u \in X - \{x, y, z\}$

Let $\bar{h} : F(X) \times F(X) \to F(X) \times F(X)$, be the homomorphism induced coordinatewise by h. Since $((x, y), (x, z)) \in \ker \pi_1 = \theta^A([\vec{C}, \vec{C}], [\vec{C}, \vec{D}])$, Corollary 3 says that

$((p, q), (p, r)) \in \theta^A\left([\vec{C}(w, ..., w), \vec{C}(w..., w)], [\vec{C}(w, ..., w), \vec{D}(w, ..., w)]\right)$

and hence we have proved (*). If we take

$\vec{0}(w) = (C_1(w, ..., w), \ldots, C_M(w, ..., w), D_1(w, ..., w), \ldots, D_M(w, ..., w))$

$\vec{1}(w) = (D_1(w, ..., w), \ldots, D_M(w, ..., w), C_1(w, ..., w), \ldots, C_M(w, ..., w))$

we may readily verify that

$\ker \pi_1 = \theta^A([\vec{0}(w), \vec{0}(w)], [\vec{0}(w), \vec{1}(w)])$

$\ker \pi_2 = \theta^A([\vec{1}(w), \vec{1}(w)], [\vec{0}(w), \vec{1}(w)])$. ■

If $\lambda \in A \in \mathcal{V}$ then we say that $\vec{e} \in A^N$ is a λ-*central element* of A if there exists an isomorphism $A \to A_1 \times A_2$ such that

$\lambda \to (\lambda_1, \lambda_2)$

$\vec{e} \to [\vec{0}(\lambda_1), \vec{1}(\lambda_2)]$.

We will use $Z_\lambda(A)$ to denote the set of λ-central elements of A.

We use ∇^A and Δ^A to denote the universal congruence on A and the trivial congruence on A, respectively. The key property of central elements is the following immediate consequence of Lemma 4

Lemma 5. *Let $\lambda \in A \in \mathcal{V}$. The map $\vec{e} \to \left(\theta^A(\vec{0}(\lambda), \vec{e}\,), \theta^A(\vec{1}(\lambda), \vec{e}\,)\right)$ is a bijection between $Z_\lambda(A)$ and the set of pairs of complementary factor congruences of A, $\{(\theta_0, \theta_1) : \theta_0 \circ \theta_1 = \nabla^A$ and $\theta_0 \cap \theta_1 = \Delta^A\}$.* ■

Our next step is to obtain an adequate first order description of the kernels of the projections associated with a central element

Lemma 6. *There exist terms P_i, $i = 1, \ldots, n$, $\vec{U} = (U_1, ..., U_m)$, $\vec{V} = (V_1, ..., V_m)$, Q_i, $i = 1, \ldots, k$, $\vec{S} = (S_1, \ldots, S_m)$ and $\vec{T} = (T_1, \ldots, T_m)$ such that the following identities hold in $\mathcal{V}$*

$$\begin{aligned}
x &\approx P_i(\vec{0}(w), \vec{U}(w, x)), \;\; i = 1, ..., n \\
x &\approx P_1(\vec{0}(w), \vec{V}(w, x, y)) \\
P_i(\vec{1}(w), \vec{V}(w, x, y)) &\approx P_{i+1}(\vec{1}(w), \vec{V}(w, x, y)), \;\; i \text{ odd} \\
P_i(\vec{0}(w), \vec{V}(w, x, y)) &\approx P_{i+1}(\vec{0}(w), \vec{V}(w, x, y)), \;\; i \text{ even} \\
P_n(\vec{1}(w), \vec{V}(w, x, y)) &\approx y,
\end{aligned}$$

$$\begin{aligned}
x &\approx Q_i(\vec{1}(w), \vec{S}(w, x)), \;\; i = 1, ..., k \\
x &\approx Q_1(\vec{1}(w), \vec{T}(w, x, y)), \\
Q_i(\vec{0}(w), \vec{T}(w, x, y)) &\approx Q_{i+1}(\vec{0}(w), \vec{T}(w, x, y)), \;\; i \text{ odd} \\
Q_i(\vec{1}(w), \vec{T}(w, x, y)) &\approx Q_{i+1}(\vec{1}(w), \vec{T}(w, x, y)), \;\; i \text{ even} \\
Q_k(\vec{0}(w), \vec{T}(w, x, y)) &\approx y.
\end{aligned}$$

Proof: By Lemma 4 we have that
$((x, x), (x, y)) \in \theta^{F(x,w) \times F(x,y,w)}([\vec{0}(w), \vec{0}(w)], [\vec{0}(w), \vec{1}(w)])$
$((x, x), (y, x)) \in \theta^{F(x,y,w) \times F(x,w)}([\vec{1}(w), \vec{1}(w)], [\vec{0}(w), \vec{1}(w)])$.
Now the terms can be obtained applying Lemma 2. ■

Let $L(w, x, y, \vec{z}, \vec{x})$ be the formula

$$x = P_1(\vec{0}, \vec{x}) \quad \wedge \left(\bigwedge_{1 \le i \le k-1, \; i \text{ odd}} P_i(\vec{z}, \vec{x}) = P_{i+1}(\vec{z}, \vec{x}) \right) \wedge$$
$$\wedge \left(\bigwedge_{1 \le i \le k-1, \; i \text{ even}} P_i(\vec{0}, \vec{x}) = P_{i+1}(\vec{0}, \vec{x}) \right) \wedge \quad P_n(\vec{z}, \vec{x}) = y$$

Let $R(w, x, y, \vec{z}, \vec{x})$ be the formula

$$x = Q_1(\vec{1},\vec{x}) \quad \wedge \left(\bigwedge_{1\le i\le k-1,\ i\ odd} Q_i(\vec{z},\vec{x}) = Q_{i+}(\vec{z},\vec{x}) \right) \wedge$$
$$\wedge \left(\bigwedge_{1\le i\le k-1,\ i\ even} Q_i(\vec{1},\vec{x}) = Q_{i+1}(\vec{1},\vec{x}) \right) \wedge \quad Q_k(\vec{z},\vec{x}) = y$$

In order to make formulas more readable we will write $R_w(x,y,\vec{z},\vec{x})$, $\vec{0}_w$, etc. in place of $R(w,x,y,\vec{z},\vec{x})$, $\vec{0}(w)$, etc.

Lemma 7. *Let $\lambda \in A \in \mathcal{V}$. For $\vec{e} \in Z_\lambda(A)$ we have*
(a) $(a,b) \in \theta^A(\vec{0}_\lambda,\vec{e})$ iff $A \models \exists\vec{x}\ L_\lambda(a,b,\vec{e},\vec{x})$
(b) $(a,b) \in \theta^A(\vec{1}_\lambda,\vec{e})$ iff $A \models \exists\vec{x}\ R_\lambda(a,b,\vec{e},\vec{x})$

Proof: (a) Suppose $(a,b) \in \theta(\vec{0}_\lambda,\vec{e})$. For $i = 1,...,N$, let $c_i \in A$ be such that
$(c_i, U_i(\lambda,a)) \in \theta(\vec{0}_\lambda,\vec{e})$
$(c_i, V_i(\lambda,a,b)) \in \theta(\vec{1}_\lambda,\vec{e})$
The reader can use Lemma 6 to see that $L_\lambda(a,b,\vec{e},\vec{c})$ holds in A, modulo $\theta(\vec{0}_\lambda,\vec{e})$ and modulo $\theta(\vec{1}_\lambda,\vec{e})$, hence $A \models \exists\vec{x}\ L_\lambda(a,b,\vec{e},\vec{x})$.

The converse follows from Lemma 2. ■

With the aid of the above lemma, it is easy to axiomatize the property "$\vec{e}$ is λ-central", by a set of first order sentences expressing that the relations $\exists\vec{x}L_\lambda(x,y,\vec{e},\vec{x})$ and $\exists\vec{x}\ R_\lambda(x,y,\vec{e},\vec{x})$ define the pair of complementary factor congruences associated with $\vec{e}$. However, these axioms will become more powerful if we construct them in such a manner that they are (obviously) preserved by direct factors [4]. Define:

$$CAN(w,\vec{z}) = \textstyle\bigwedge_{i=1}^{N} \left(\exists\vec{u}_i\ L_w(0_{iw}, z_i, \vec{z}, \vec{u}_i) \wedge \exists\vec{v}_i\ R_w(1_{iw}, z_i, \vec{z}, \vec{v}_i) \right)$$
$$PROD(w,\vec{z}) = \forall x,y \exists z,\vec{u},\vec{v}\ \left(L_w(x,z,\vec{z},\vec{u}) \wedge R_w(z,y,\vec{z},\vec{v}) \right)$$
$$INT(w,\vec{z}) = \forall x,y,\vec{x},\vec{v}\ \left(L_w(x,y,\vec{z},\vec{x}) \wedge R_w(x,y,\vec{z},\vec{v}) \to x = y \right)$$
$$REF^L(w,\vec{z}) = \forall x \exists\vec{x}\ L_w(x,x,\vec{z},\vec{x})$$

$$SYM^L(w,\vec{z}) = \forall x,y,z,\vec{x},\vec{y},\vec{v}\ \Big(L_w(x,y,\vec{z},\vec{x}) \wedge L_w(y,z,\vec{z},\vec{y}) \wedge R_w(z,x,\vec{z},\vec{v}) \to$$
$$\to z = x \Big)$$

$$TRANS^L(w,\vec{z}) = \forall x,y,z,u,\vec{x},\vec{y},\vec{u},\vec{v}\ \Big(L_w(x,y,\vec{z},\vec{x}) \wedge L_w(y,z,\vec{z},\vec{y}) \wedge$$
$$\wedge L_w(x,u,\vec{z},\vec{u}) \wedge R_w(u,z,\vec{z},\vec{v}) \to u = z \Big)$$

For each n-ary function symbol f, define:

$$PRES_f^L(w,\vec{z}) = \forall x_1,y_1,...,x_n,y_n,z,\vec{x}_1,...,\vec{x}_n,\vec{u},\vec{v}$$
$$\Big(\bigwedge_j L_w(x_j,y_j,\vec{z},\vec{x}_j) \ \wedge L_w(f(x_1,\dots,x_n),z,\vec{z},\vec{u}) \wedge R_w(z,f(y_1,\dots,y_n),\vec{z},\vec{v}) \to$$
$$\to z = f(y_1,...,y_n) \Big)$$

Finally, define REF^R, SYM^R, $TRANS^R$ and $PRES_f^R$ to be the result of interchanging L with R in REF^L, SYM^L, $TRANS^L$ and $PRES_f^L$, respectively, and let Σ be the union of the following two sets
$\{CAN, PROD, INT, REF^L, SYM^L, TRANS^L, REF^R, SYM^R, TRANS^R\}$
$\{PRES_f^L, PRES_f^R : f \text{ a function symbol}\}$.

Lemma 8. *For every* $\lambda \in A \in \mathcal{V}$, $\vec{e} \in A^N$,

$$\vec{e} \in Z_\lambda(A) \text{ iff } A \models \sigma(\lambda, \vec{e}), \text{ for every } \sigma \in \Sigma.$$

Proof: Define the following binary relations on A:

$$\theta_\lambda = \{(x, y) \in A^2 : A \models \exists \vec{x}\ L_\lambda(x, y, \vec{e}, \vec{x})\}$$

$$\delta_\lambda = \{(x, y) \in A^2 : A \models \exists \vec{x}\ R_\lambda(x, y, \vec{e}, \vec{x})\}$$

($\Rightarrow$) By Lemmas 4 and 7 we can suppose $A = A_0 \times A_1$ and $\vec{e} = [\vec{0}, \vec{1}]$ with θ_λ and δ_λ the kernels of the canonical projections. Now, the axioms Σ are obvious.

($\Leftarrow$) It will follow from the axiom set Σ that θ_λ and δ_λ are a pair of complementary factor congruences of A. Note that θ_λ is reflexive since $A \models REF^L(\lambda, \vec{e})$. To see that θ_λ is transitive let's suppose that $(x, y) \in \theta_\lambda$ and $(y, z) \in \theta_\lambda$. Then, there exist $\vec{x}, \vec{y}$ such that
$A \models L_\lambda(x, y, \vec{e}, \vec{x}) \wedge L_\lambda(y, z, \vec{e}, \vec{y})$.
Since $A \models PROD(\lambda, \vec{e})$, there exist $u, \vec{u}, \vec{v}$ such that
$A \models L_\lambda(x, u, \vec{e}, \vec{u}) \wedge R_\lambda(u, z, \vec{e}, \vec{v})$.
Since $A \models TRANS^L(\lambda, \vec{e})$ we obtain $u = z$ and hence $(x, z) \in \theta_\lambda$. By similar considerations it can be proved that θ_λ is symmetric and preserves the fundamental operations of A. Thus we can assume that θ_λ and δ_λ are congruences on A. To see that they are factor complementary note that $\theta_\lambda \circ \delta_\lambda = \nabla^A$ follows from $A \models PROD(\lambda, \vec{e})$ and that $\theta_\lambda \cap \delta_\lambda = \Delta^A$ follows from $A \models INT(\lambda, \vec{e})$. We conclude the proof noting that the canonical isomorphism $A \to A/\theta_\lambda \times A/\delta_\lambda$ carries $\vec{e}$ into $[\vec{0}_\lambda, \vec{1}_\lambda]$, since $A \models CAN(\lambda, \vec{e})$. ■

Corollary 9. *For every* $(\lambda_1, \lambda_2) \in A_1 \times A_2 \in \mathcal{V}$, $\vec{e} \in A_1^N$, $\vec{f} \in A_2^N$,

$$[\vec{e}, \vec{f}] \in Z_{(\lambda_1, \lambda_2)}(A_1 \times A_2) \text{ iff } \vec{e} \in Z_{\lambda_1}(A_1) \text{ and } \vec{f} \in Z_{\lambda_2}(A_2)$$

Proof: ($\Rightarrow$). Note that CAN, $PROD$, REF^L, REF^R are positive formulas and hence they are preserved by direct factors. Finally note that the remainder of axioms in Σ are of the form $\forall \vec{x}\ (\varphi(w, \vec{z}, \vec{x}) \to x_i = x_j)$ with $REF^L(w, \vec{z}) \to \exists \vec{x}\ \varphi(w, \vec{z}, \vec{x})$ universally valid, and since $\forall \vec{x}\ (\varphi(w, \vec{z}, \vec{x}) \to x_i = x_j) \wedge \exists \vec{x}\ \varphi(w, \vec{z}, \vec{x})$ is preserved by direct factors, we have the result.

($\Leftarrow$). The axioms of Σ are Horn formulas and hence they are preserved by direct products. ■

Corollary 10. *If $\mathcal{V}$ is a variety with CFC over a finite language, then the class of directly indecomposable members of $\mathcal{V}$ is axiomatizable by a finite set of $\forall\exists\forall$-sentences.*

Proof: Note that $A \in \mathcal{V}$ is directly indecomposable iff
$\forall w \forall \vec{z} \left(\vec{z} \in Z_w(A) \to \vec{z} = \vec{0}_w \vee \vec{z} = \vec{1}_w \right)$. ■

Proof of Theorem 1: Let θ_0 and θ_1 be a pair of complementary factor congruences of A and let θ be a factor congruence on A. We will prove that $(\theta_0 \vee \theta) \cap (\theta_1 \vee \theta) = \theta$, which by [1] implies BFC. Let $\lambda \in A$. By Lemma 5 there exists a λ-central element $\vec{e}$ such that $\theta_0 = \theta^A(\vec{0}_\lambda, \vec{e})$ and $\theta_1 = \theta^A(\vec{1}_\lambda, \vec{e})$. By Corollary 9, we have that $\vec{e}/\theta \in Z_{\lambda/\theta}(A/\theta)$ and hence

$$\theta^{A/\theta}(\vec{0}_{\lambda/\theta}, \vec{e}/\theta) \cap \theta^{A/\theta}(\vec{0}_{\lambda/\theta}, \vec{e}/\theta) = \Delta^{A/\theta}$$

Thus we have

$$(\theta^A(\vec{0}_\lambda, \vec{e}) \vee \theta)/\theta \cap (\theta^A(\vec{1}_\lambda, \vec{e}) \vee \theta)/\theta = \theta/\theta$$

which implies $(\theta_0 \vee \theta) \cap (\theta_1 \vee \theta) = \theta$. ■

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